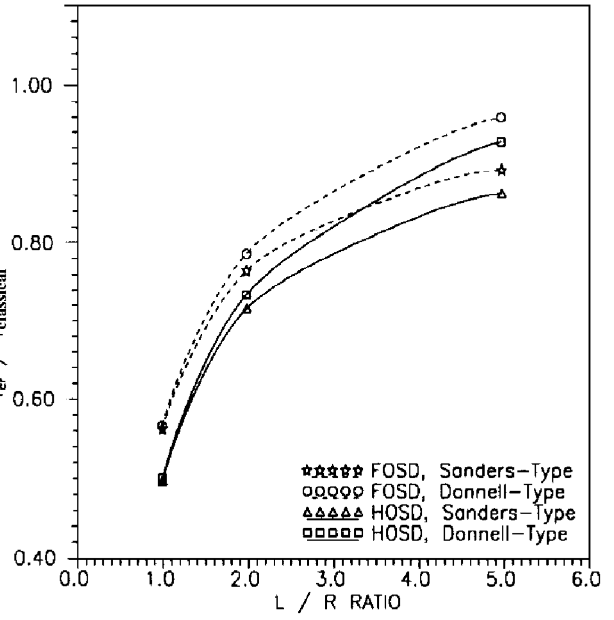

Fig. 8 Normalized critical torsional load for (45, 45, _45)$_s$ deg, *R*/*h* = 15.

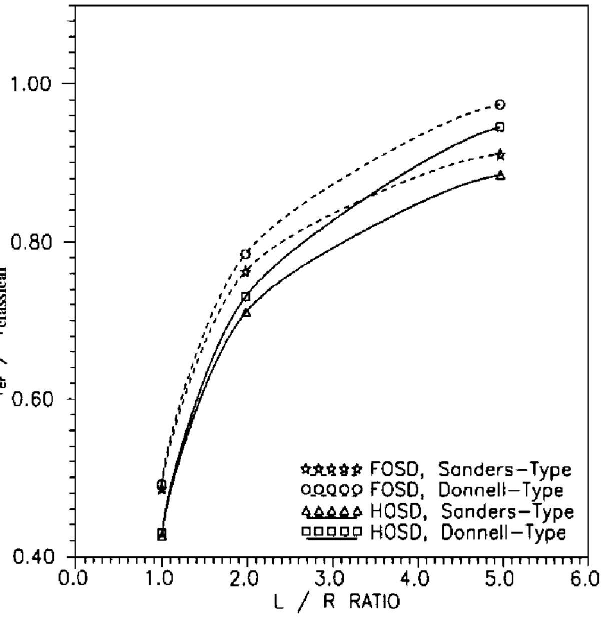

Fig. 9 Normalized critical torsional load for (30, 30, _60)$_s$ deg, *R*/*h* = 15.

Figures 1–9 show the critical torsional load for the shear deformation theories normalized with respect to the values obtained by the classical theory. In general, critical loads obtained by employing Donnell-type relations predicted higher values than those obtained by employing Sanders-type relations. This observation is true for all R/h ratios and L/R ratios. The difference between the two sets is small for small L/R and R/h ratios and becomes more evident as the shell becomes thicker and longer. The discrepancy between critical loads obtained from the two different approximations (Sanders and Donnell) is primarily affected by L/R ratio and to lesser degree by R/h ratio. When n (the number of full waves) is > 3, the two sets yield almost the same critical load (within 1%), but for $n \leq 3$ the computed difference can be as large as 8%. For all of the generated results (all R/h and L/R ratios), the critical loads were within the range of 0.5–8% above the ones predicted by employing Sanders-type relations. This is attributed to the fact that the rotation $[\bar{v}/(R+z)]$ has more effect when the number of circumferential waves n is < 3. Note that the effect of L/R ratio is the same for all three theories and including the effect of shear deformation has no effect on the behavior of the critical loads obtained by Donnell-type relations.

Acknowledgment

Computing support was provided by the Ohio Supercomputer Center and it is gratefully acknowledged.

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Final Solution of Duffing Equation of Mixed Parity

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Introduction

HIAMANG and Mickens[1] have examined the periodic solution of the nonlinear second-order differential equation of motion,

$$\ddot{x} + x^3 = 0 \tag{1}$$

with the initial conditions

$$x = x_0, \qquad \dot{x} = 0 \qquad \text{at} \qquad \tau = 0 \tag{2}$$

and the corresponding first-order differential-equation energy relation, applying the method of harmonic balance with lower-order harmonics. The overdots denote differentiation with respect to time τ. The values of angular frequency determined from the equation of motion and the energy relation differ by approximately 20%. The periodic solution from the equation of motion is found to be an excellent first approximation to the exact solution. On the basis of these observations, they concluded that the application of harmonic balance, in the lowest-order approximation, to the energy relation is not particularly helpful in obtaining the periodic solution and suggested using only the equation of motion. Strictly speaking, the first-order differential-equation energy relation is derived from the second-order differential equation of motion, and so, one should expect the same exact/approximate result from these equations. Reinvestigation of this problem by Narayana Murty and Nageswara Rao[2] reveals that the solution from the Hiamang–Mickens[1] energy equation does not satisfy the initial conditions exactly. The inclusion of higher-order harmonics in the method of harmonic balance gives better

Received Oct. 7, 1996; accepted for publication Nov. 14, 1996.

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agreement from the equation of motion (1) and the corresponding energy relation:

$$(\dot{x})^2 = \tfrac{1}{2}(x_0^4 - x^4) \tag{3}$$

To investigate these issues further, a similar nonlinear equation as suggested by Mickens,[3]

$$\ddot{x} + x^3(1 + \lambda x^2)^{-1} = 0, \qquad \lambda > 0 \tag{4}$$

is examined by Radhakrishnan et al.[4] It is found that the discrepancy is due to the singularity in $\dot{x}$ (at $\tau = 0$) creeping into the energy relation. In the previous studies, the restoring force function in the equation of motion is an odd function and the behavior of oscillations is the same for both negative and positive amplitudes.

For nonlinear free vibration of laminated anisotropic plates,[5] the restoring force function in the equation of motion is found to be a cubic polynomial, which is in the form of a Duffing-type equation or a combination of quadratic and cubic terms. For a nonodd restoring force function, the behavior of oscillations is different for positive and negative amplitudes. Hence, it is necessary and useful for all practical purposes to examine the uniqueness of angular frequency from the equation of motion and its energy relation, using the harmonic balance with lower-order harmonics.

Analysis

The second-order nonlinear differential (Duffing-type) equation is in the form

$$\ddot{x} + f(x) = 0 \tag{5}$$

The restoring force function,

$$f(x) = \alpha x + \beta x^2 + \gamma x^3 \tag{6}$$

in the equation of motion (5) is a nonodd function for $\beta \neq 0$. Here α, β, and γ are real constants. The behavior of oscillations is different for positive and negative amplitudes (x_+ and x_-), which can be found by equating the potential energies in either position, i.e., from

$$I(x_-) = I(x_+) \tag{7}$$

where

$$I(x) = \int_0^x f(y)\,\mathrm{d}y = \frac{1}{2}x^2\left(\alpha + \frac{2}{3}\beta x + \frac{1}{2}\gamma x^2\right)$$

Multiplying Eq. (5) by $\dot{x}$ and using the initial conditions

$$x = x_+, \qquad \dot{x} = 0 \qquad \text{at} \qquad \tau = 0 \tag{8}$$

after integration, one obtains the energy relation

$$(\dot{x})^2 = 2[I(x_+) - I(x)] = (x_+ - x)(x - x_-)(c_0 + c_1 x + c_2 x^2) \tag{9}$$

where

$$c_0 = \alpha + \tfrac{4}{3}\beta a + \tfrac{1}{2}\gamma(3a^2 + b^2), \qquad c_1 = \tfrac{2}{3}\beta + \gamma a$$

$$c_2 = \tfrac{1}{2}\gamma, \qquad a = \tfrac{1}{2}(x_+ + x_-), \qquad b = \tfrac{1}{2}(x_+ - x_-)$$

The negative amplitude x_- corresponding to the positive amplitude x_+ obtained from Eq. (7) is

$$x_- = \frac{S_1 + S_2 - a_2}{a_3} \tag{10}$$

where

$$S_{1,2} = \left[r \pm (q^3 + r^2)^{\frac{1}{2}}\right]^{\frac{1}{3}}, \qquad q = a_3 a_1 - a_2^2$$

$$r = \tfrac{1}{2}(3a_3a_2a_1 - a_3^2 a_0) - a_2^3, \qquad a_3 = c_2$$

$$a_2 = \tfrac{1}{3}\left(\tfrac{2}{3}\beta + a_3 x_+\right), \qquad a_1 = \tfrac{1}{3}\alpha + a_2 x_+, \qquad a_0 = 3a_1 x_+$$

From Eqs. (8) and (9), one obtains

$$\int_{x_-}^{x_+} \frac{\mathrm{d}x}{\sqrt{2[I(x_+) - I(x)]}} = \int_0^{T/2} \mathrm{d}\tau = \frac{T}{2} = \frac{\pi}{\omega} \tag{11}$$

Table 1 Comparison of angular frequency ω for the specified positive amplitude x_+

Amplitudes x_+	Amplitudes x_-	Angular frequency, ω: Exact integration, Eq. (13)	Harmonic balance method, Eq. (16) or (18)	Others
Case 1) $\alpha = 0$, $\beta = 0$, $\gamma = 1$				
1.0	−1.0	0.8472	0.8660	0.8472[a]
2.0	−2.0	1.6944	1.7321	1.6944[a]
3.0	−3.0	2.5417	2.5981	2.5416[a]
Case 2) $\alpha = 0$, $\beta = -1$, $\gamma = 1$				
$1\frac{5}{6}$	−1.0842	1.0848	1.1260	1.1637[b]
$3\frac{1}{3}$	−2.6495	2.4645	2.5255	2.5331[b]
$6\frac{1}{3}$	−5.6625	5.0471	5.1622	5.1640[b]
Case 3) $\alpha = 1$, $\beta = -2.2518$, $\gamma = 2.54328$				
0.3	−0.2089	0.9448	0.9666	0.9325[c]
1.2	−0.7308	1.4376	1.4633	1.4373[c]
3.0	−2.4270	3.7138	3.7922	3.7883[c]

[a]Ref. 1. [b]Ref. 6. [c]Ref. 5.

where T is the period and ω is the angular frequency. For the specified positive amplitude (x_+), the corresponding negative amplitude (x_-) can be obtained directly from Eq. (10). By substituting x_- and x_+ in Eq. (11), one can obtain the angular frequency ω. The integrand in Eq. (11) has poles at the endpoints of integration, i.e., at $x = x_-$ and x_+, which may adversely affect the accuracy of an integration rule. Hence, the integrand in Eq. (11) is modified by using a transformation

$$x = a + b\cos[(\pi/2)(1 + \xi)] \tag{12}$$

that eliminates the singularities and yields a form

$$\omega = \left(\frac{1}{2}\int_{-1}^{1} \frac{\mathrm{d}\xi}{\sqrt{c_0 + c_1 x + c_2 x^2}}\right)^{-1} \tag{13}$$

that can be evaluated quite easily by quadratures. A 10-point Gaussian rule was adopted here for evaluating the integral.

For the lowest-order harmonic, the periodic solution that satisfies the initial conditions (8) is

$$x = a + b\cos(\omega\tau) \tag{14}$$

Substituting Eq. (14) in Eq. (5) and neglecting the higher-order harmonics, one obtains

$$\alpha a + \beta\left(a^2 + \tfrac{1}{2}b^2\right) + \gamma a\left(a^2 + \tfrac{3}{2}b^2\right) = 0 \tag{15}$$

$$\omega^2 = \alpha + 2\beta a + \tfrac{3}{4}\gamma(4a^2 + b^2) \tag{16}$$

Because $(x_+ - x)$ and $(x - x_-)$ are the factors on the right-hand side of the first-order differential equation energy relation (9), $\dot{x}$ becomes zero when x is x_+ or x_-. This is the reason that the integrand in Eq. (11) shows poles at $x = x_+$ and x_- as the energy relation is integrated from $\tau = 0$ to $T/2$. The energy equation (9) can be written in the form

$$\frac{(\dot{x})^2}{(x_+ - x)(x - x_-)} = c_0 + c_1 x + c_2 x^2 \tag{17}$$

Substituting Eq. (14) in Eq. (17) and neglecting the higher-order harmonics, one obtains the angular frequency ω as

$$\omega^2 = c_0 + c_1 a + c_2\left(a^2 + \tfrac{1}{2}b^2\right) \tag{18}$$

which is found to be identical to Eq. (16). As one should expect, the energy relation being the first integral of the equation of motion, the two procedures resulted in exactly the same solution.

Validity of the Solution

To verify the adequacy of the present harmonic balance method with lower-order harmonics, the periodic solution of the Duffing-type equation (5) is examined for the following three cases having the values of α, β, and γ:

Case 1) $\alpha = 0, \beta = 0, \gamma = 1$ (Hiamang and Mickens[1])
Case 2) $\alpha = 0, \beta = -1, \gamma = 1$ (Gottlieb[6])
Case 3) $\alpha = 1, \beta = -2.2518, \gamma = 2.54328$ (Nageswara Rao[5])

The angular frequencies ω for the specified values of the positive amplitude x_+ are presented in Table 1.

Conclusions

The harmonic balance method with lower-order harmonics yields good results that are comparable with those obtained by exact integration. The frequency–amplitude relation obtained for the Duffing-type equation using the method of harmonic balance with lower-order harmonics will be useful for all practical purposes in understanding the nonlinear free-vibration characteristics of composite structures.

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Singularities in Polynomial Representations of Transverse Shear in Finite Elements

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I. Introduction

POLYNOMIALS are frequently used to represent transverse shear when modeling composite shells with two-dimensional finite elements. Layerwise theories have been developed that capture the continuous transverse shear stresses and displacements through the thickness of the laminate, but such schemes do not permit the use of complete cubic polynomials in representing the displacement functions: the number of unknown coefficients exceeds the number of conditions imposed on the laminate. Therefore, choices must be made regarding the best representation of an incomplete cubic, and only a few choices make physical sense. Of these, it is shown that (at least) one choice for these polynomials exists that can produce ill-behaved (even singular) displacement functions.

II. Theory

Use of higher-order shear theories (HSTs) to represent transverse shear behavior usually involves representing the in-plane displacements u_i, $i \in \{1, 2\}$, as a function of the laminate thickness coordinate z, by polynomials having unknown coefficients. Moreover, one may choose to represent displacements in each ply of the laminate,[1] such that the displacements within the kth ply of a laminate have the form

$$u_1^{(k)}(z) = \left(a_{14}^{(k)} + b_{14}^{(k)} z + c_{14}^{(k)} z^2 + d_{14}^{(k)} z^3\right)\gamma_4 + \left(a_{15}^{(k)} + b_{15}^{(k)} z + c_{15}^{(k)} z^2 + d_{15}^{(k)} z^3\right)\gamma_5 \tag{1a}$$

$$u_2^{(k)}(z) = \left(a_{24}^{(k)} + b_{24}^{(k)} z + c_{24}^{(k)} z^2 + d_{24}^{(k)} z^3\right)\gamma_4 + \left(a_{25}^{(k)} + b_{25}^{(k)} z + c_{25}^{(k)} z^2 + d_{25}^{(k)} z^3\right)\gamma_5 \tag{1b}$$

where the angles γ_4 and γ_5 represent local shear rotation in the 2–3 (y–z) and 1–3 (x–z) planes, respectively. To find these displacements, one must find $16N$ unknown coefficients (where N is the total number of plies in the laminate), $a_{mn}^{(k)}$, $b_{mn}^{(k)}$, $c_{mn}^{(k)}$, and $d_{mn}^{(k)}$.

Because continuity of interlaminar stresses and displacements do not provide all of the $16N$ equations necessary, the displacement functions must be simplified by omitting coefficients.

The theory presented by Pai and Palazotto[2] incorporates a local and layerwise displacement theory using the Jaumann (or Biot–Cauchy–Jaumann) strains B_{mn} and stresses J_{mn}. Jaumann measures are equivalent to local engineering measures and, hence, are suitable for analyzing large deformation and small strains.

By enforcing shear stress continuity and in-plane displacement continuity at each ply interface, as well as assuming a shear strain-free condition on the exterior surfaces, the resulting $4N$ algebraic equations can be stated as

$$B_{23}^{(1)}(x, y, z_1) = 0, \qquad B_{13}^{(1)}(x, y, z_1) = 0 \tag{2a}$$

$$u_1^{(k)}(x, y, z_{k+1}) - u_1^{(k+1)}(x, y, z_{k+1}) = 0 \quad \text{for} \quad k = 1, \ldots, N-1 \tag{2b}$$

$$u_2^{(k)}(x, y, z_{k+1}) - u_2^{(k+1)}(x, y, z_{k+1}) = 0 \quad \text{for} \quad k = 1, \ldots, N-1 \tag{2c}$$

$$J_{23}^{(k)}(x, y, z_{k+1}) - J_{23}^{(k+1)}(x, y, z_{k+1}) = 0 \quad \text{for} \quad k = 1, \ldots, N-1 \tag{2d}$$

$$J_{13}^{(k)}(x, y, z_{k+1}) - J_{13}^{(k+1)}(x, y, z_{k+1}) = 0 \quad \text{for} \quad k = 1, \ldots, N-1 \tag{2e}$$

$$B_{23}^{(N)}(x, y, z_{N+1}) = 0, \qquad B_{13}^{(N)}(x, y, z_{N+1}) = 0 \tag{2f}$$

These $4N$ equations become $8N$ equations when we insist that they be satisfied for γ_4 and γ_5 independently, but this is still short of the requisite $16N$ needed by Eqs. (1). Pai and Palazotto[2] suggested the following incomplete cubic forms for the displacement functions:

$$\begin{aligned} g_{14}^{(k)} &\equiv c_{14}^{(k)} z^2 + d_{14}^{(k)} z^3, & g_{15}^{(k)} &\equiv z + c_{15}^{(k)} z^2 + d_{15}^{(k)} z^3 \\ g_{24}^{(k)} &\equiv z + c_{24}^{(k)} z^2 + d_{24}^{(k)} z^3, & g_{25}^{(k)} &\equiv c_{25}^{(k)} z^2 + d_{25}^{(k)} z^3 \end{aligned} \tag{3}$$

While using the functions of Eqs. (3) in the context of studying the elasticity solutions of Pagano,[3] it was found that certain choices of shear moduli and ply thicknesses of a [0/90/0] laminated plate strip could induce artificially high shear stiffness.

Received April 2, 1996; revision received March 12, 1997; accepted for publication March 25, 1997.

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